Project Work Optical And Microwave Propagation

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5th topic

Propagation in dielectric rod

An illustrated group work for the course optical and microwave propagation at the University of Technology Chemnitz, summer 2001.

Contents

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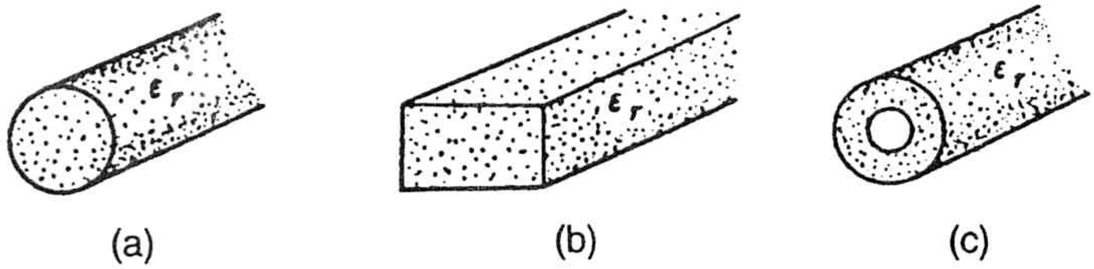
Introduction:

For propagation of a electromagnetic wave in a coaxial wave guide metal walls were required, e.g. a inner conductor, the solid hull, or the metal cover of a (for example cylindrical) wave guide.
Additional, the electromagnetic wave is able to propagate in a dielectric rod as shown in Fig. 1. This is called the dielectric rod guide.



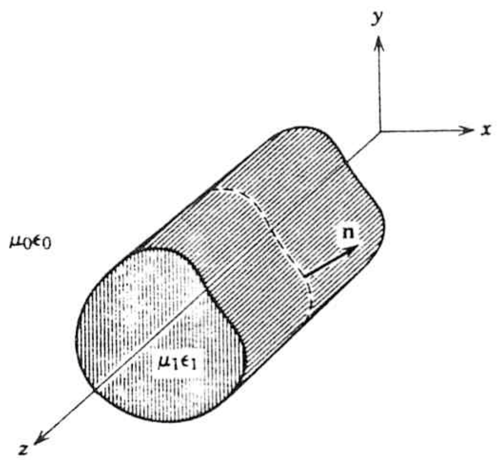
Pic. 1 [2] – the dielectric rod guide (a – cylindrical, b – rectangular, c – tube)

In this dielectric rod, the electromagnetic wave cannot disappear suddenly at the end of the rod, but decreases exponentially on the outside of the rod along the axis in the region of air.

Conditions:

In this draft, we use a solid cylinder for explaining the propagation of a electromagnetic wave in a dielectric rod.

A dielectric cylinder can be used as a wave guide. It has nearly the same characteristics like a hollow metal cylinder, if the dielectric constant ε of the dielectric cylinder is high enough.
However, there are characteristic differences between a dielectric rod guide and a cylindrical wave guide. The reason for the differences are the different edge conditions, which must be fulfilled by the fields on the surface of the cylinder.
The surface of a cylinder with any cross-section shows Pic. 2. For simplifying the conditions, we assume that the surface and the cross-section are constant along the cylinder axis.



Pic. 2 [2] – the dielectric wave guide

We assume also that the dielectric solid and the medium surrounding the solid is homogeneous and not magnetic. The dielectric solid has the relative permittivity ε_1 and the relative permeability μ_1, the outer medium has the relative permittivity ε_0 and the relative permeability μ_0.

The Maxwell Equations

For describing the fields in the dielectric solid we use the Maxwell equations (James Clerk Maxwell, 1831 - 1879, Physician).

(1) We assume that $\vec{E}$ and $\vec{B}$ are harmonic in time:

$$\vec{E} = \vec{E}_0 e^{-j\omega t}$$
$$\vec{B} = \vec{B}_0 e^{-j\omega t}$$

(2) With (1), Maxwell's equations for our problem are:

$$\nabla \times \vec{E} = j\omega\vec{B} \qquad \nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{B} = -j\mu\varepsilon\omega\vec{E} \qquad \nabla \cdot \vec{E} = 0$$

Solving the Maxwell Equations

(3) We can solve the equations (2) to:

$$\mu\varepsilon\omega^2\vec{E} + \nabla^2\vec{E} = 0$$
$$\mu\varepsilon\omega^2\vec{B} + \nabla^2\vec{B} = 0$$

(4) Throwing in cylindrical symmetry of our problem we can separate the change of the fields in the direction of z:

$$\vec{E}(x,y,z,t) = \vec{E}(r,\varphi)e^{jkz-j\omega t}$$
$$\vec{B}(x,y,z,t) = \vec{B}(r,\varphi)e^{jkz-j\omega t}$$

The value of k (wave number) is currently unknown.

(5) We assume that the propagation of the fields depends only on the direction of z:

$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\vec{E} + \left(\mu\varepsilon\omega^2 - k^2\right)\vec{E} = 0$$
$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\vec{B} + \left(\mu\varepsilon\omega^2 - k^2\right)\vec{B} = 0$$

(6) Now we can transform the equations into cylindrical coordinates:

$$\mu\varepsilon\omega\ E_z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \gamma^2 k E_z = 0$$

$$\mu\varepsilon\omega\ B_z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 B_z}{\partial \varphi^2} + ß^2 k B_z = 0$$

(7) With the definition of the two constants:

Inner area: $\gamma^2 = \mu_1\varepsilon_1\omega^2 - k^2$

Outer area: $ß^2 = k^2 - \mu_0\varepsilon_0\omega^2$

(8) We assume that the propagation of the waves are independent from the value of φ. We take the differences between the solid and the outer medium into account and can solve (6) to:

Inner area: $$\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \gamma^2\right)\begin{pmatrix}E_z\\ B_z\end{pmatrix} = 0$$

Outer area: $$\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - ß^2\right)\begin{pmatrix}E_z\\ B_z\end{pmatrix} = 0$$

We solve the equations for the case B_z in the inner area. The solution for the case of the outer area of E_z can be found in the same way (see equation 17).

(9) We need the Bessel equations for solving and transform (8) to:

$$\left(\frac{\partial^2 B_z}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \gamma^2\right)B_z = 0 = r^2\frac{\partial^2 B_z}{\partial r^2}B_z + r\frac{\partial}{\partial r}B_z + r^2\gamma^2 B_z$$

(10) We substitute two constants, γ for the inner area and $ß$ for the outer area:

$$\gamma = \frac{x}{r} \quad \text{analog} \quad ß = \frac{x}{r}$$

(11) And use following derivations:

$$\frac{\partial^2 B_z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial B_z}{\partial x}\frac{\partial x}{\partial r}\right) = \frac{\partial x}{\partial r}\frac{\partial^2 B_z}{\partial x \partial r} + \frac{\partial B_z}{\partial x}\frac{\partial^2 B_z}{\partial r^2} = \frac{\partial^2 B_z}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 = \frac{\partial^2 B_z}{\partial x^2}\gamma^2$$

$$\frac{\partial B_z}{\partial r} = \frac{\partial B_z}{\partial x}\frac{\partial x}{\partial r} = \frac{\partial B_z}{\partial x}\gamma$$

The Bessel and the Weber equations

(12) With (11) in (9) we get the Bessel equation:

$$\frac{x^2}{\gamma^2}\gamma^2\frac{\partial^2 B_z}{\partial x^2}+\frac{x}{\gamma}\gamma\frac{\partial B_z}{\partial r}+\gamma^2\frac{(x^2-n^2)}{\gamma^2}B_z=0$$

As we can see n has the value of 0.

The universal solution for real n is:

(13.1)

$$B_z=C_1J_n(x)+C_2J_{-n}(x)$$

(13.2)

$$J_n=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(n+k+1)}$$

(13.3)

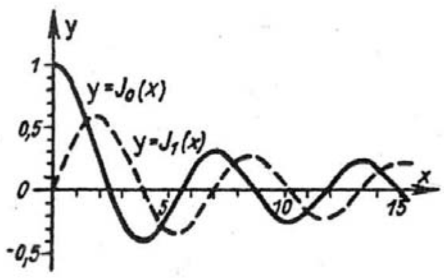
Weber function: $Y_n(x)=\lim_{m\to n}\frac{J_m(x)\cos m\pi-J_{-m}(x)}{\sin m\pi}$

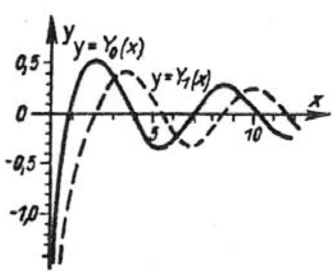
(13.4)

$$\Gamma(x)=\int_0^{\infty}e^{-t}t^{x-1}dt$$

(13.5)

universal case: $J_{-n}(x)=Y_n(x)$





Pic. 3 [1] – Graph of the Bessel function (1. sort) and of the Weber function

We demand that $|B_z(x=0)|\neq\infty$. The Weber function don't fulfill that condition so we have to set $C_2=0$. (Pic. 3)

We distinguish between the inner and the outer area:

(14) For n=0 we get:

$$B_{zINNER} = C_1 J_0(x) = C_1 J_0(r\gamma) = C_1 J_0(r\gamma)$$

(15) For the outer area we get because of (8) this Bessel equation:

$$x^2 \frac{\partial^2 B_z}{\partial x^2} + x \frac{\partial B_z}{\partial r} - (x^2 + n^2) B_z = 0$$

As we can see n has again the value of 0.

The universal solution is:

(16.1)

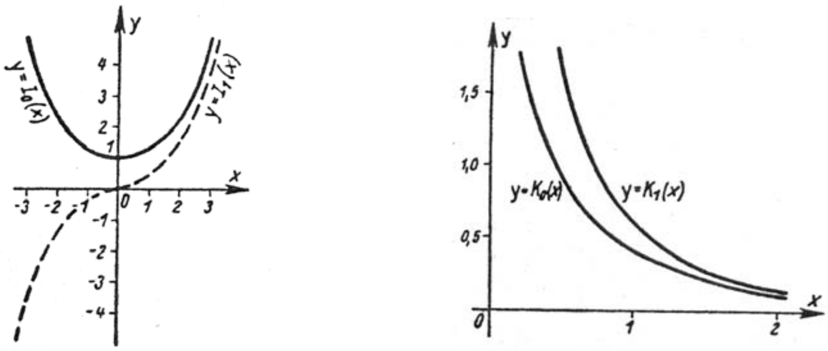
$$B_{zOUTER} = C_3 I_n(c) + C_4 K_n(x)$$

(16.2)

Macdonald function: $K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin n\pi}$

(16.3)

$$I_n(x) = j^{-n} J_n(jx)$$



Pic. 4 [1] – Graph of the imaginary Bessel Function and of the Macdonald function

We demand that $|B_z(x=\infty)| = 0$. $I_n(x)$ don't fulfill that condition and we can ignore it.

(17) For n=0 get:

$$B_{zOUTER} = C_4 K_0(x) = C_4 K_0(r\beta) = C_4 K_0(r\beta)$$

<u>General solution for the inner and outer case</u>

In analogy we can derive this for E_z, so we get the general solution

(18)

$$\begin{pmatrix} B_{zOUTER} \\ E_{zOUTER} \end{pmatrix} = C_1 K_0 (rß)$$

(19)

$$\begin{pmatrix} B_{zINNER} \\ E_{zINNER} \end{pmatrix} = C_2 J_0 (r\gamma)$$

Before solving the other components, we have to determine the wave types which can propagate. In our case this are TE (transversal electric, $E_z = 0$) and TM (transversal magnetic, $B_z = 0$) wave types. At first we will just concentrate on TE-waves and afterwards draw conclusions for the other type.

Now we can derive the other Components of our field with the given B_z and $E_z = 0$ by using the Maxwell Equations (2) in cylindrical components

(20.1)

$$j\omega B_z = \frac{1}{r}\frac{\partial}{\partial r}\left(rE_\varphi\right) - \frac{1}{r}\frac{\partial E_r}{\partial \varphi}$$

(20.2)

$$j\omega B_r = \frac{1}{r}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z}$$

(20.3)

$$j\omega B_\varphi = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}$$

(20.4)

$$-j\omega\mu\varepsilon E_z = \frac{1}{r}\frac{\partial}{\partial r}\left(rB_\varphi\right) - \frac{1}{r}\frac{\partial B_r}{\partial \varphi}$$

(20.5)

$$-j\omega\mu\varepsilon E_r = \frac{1}{r}\frac{\partial B_z}{\partial \varphi} - \frac{\partial B_\varphi}{\partial z}$$

(20.6)

$$-j\omega\mu\varepsilon E_\varphi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}$$

Special solution for the inner field

We assume the dependency from the φ-component is finite small so we can ignore it. First we search a solution for the inner field by taking (20.2) and also (20.6) and rewrite them according to (14) and (4):

(21.1)

$$j\omega B_r = -E_\varphi jk$$

(21.2)

$$-j\omega\eta\varepsilon E_\varphi = B_r jk - B_z^{'}$$

By inaugurating (21.1) in (21.2) we come to

(22)

$$B_r = \frac{jk}{j\omega}\left(\frac{B_r jk - B_z^{'}}{j\omega\mu\varepsilon}\right) = -\frac{j}{\omega^2\mu\varepsilon}(jkB_r - B_z^{'}) = \frac{jB_z^{'}}{\omega^2\mu\varepsilon - k^2}$$

At last we use (7) and arrive to

(23)

$$B_r = \frac{jk}{\gamma^2}\frac{\partial B_z}{\partial r}$$

Only the derivate of B_z is unclear, but with the solution of the Bessel/Weber Functions from [1] page 399-400 we can dissolve it to

(24)

$$\frac{\partial B_z}{\partial r} = \left(C_1 J_0(r\gamma)\right)^{'} = J_{-1}(r\gamma) = (-1)^{-1} J_1(r\gamma)$$

<u>Final description of the propagation</u>

The other components for the inner and the outer field can be analogous easily derived so we find the final description of the propagation for TE waves.

(25)

$$\left.\begin{array}{l} (1): B_z = J_0(\gamma r) \\ (2): B_r = -\dfrac{jk}{\gamma} J_1(\gamma r) \\ (3): E_\varphi = \dfrac{j\omega}{\gamma} J_1(\gamma r) \\ (4): E_z = E_r = B_\varphi = 0 \end{array}\right\} r \le a$$

$$\left.\begin{aligned}(5) &: B_z = AK_0(ßr)\\ (6) &: B_r = -\frac{jkA}{ß}K_1(ßr)\\ (7) &: E_\varphi = \frac{j\omega A}{ß}K_1(ßr)\\ (8) &: E_z = E_r = B_\varphi = 0\end{aligned}\right\} r > a$$

We define a as the value of the radius of the dielectric rod. (If $r = a$ we are on the surface of the dielectric rod.)

Our border condition for the inner and the outer field is that they have to be equal for $r = a$, so we have to set

(27.1)

$$C_1 J_0(a\gamma) = C_2 K_0(a\beta)$$

(27.2)

$$\frac{C_1 J_1(a\gamma)}{\gamma} = -\frac{C_2 K_1(a\beta)}{\beta}$$

By eliminating the two constants we can derive the determination equations for β and γ, if we throw in and rewrite (7)

(28.1)

$$\frac{J_1(a\gamma)}{\gamma J_0(a\gamma)} + \frac{K_1(a\beta)}{\beta K_0(a\beta)} = 0$$

(28.2)

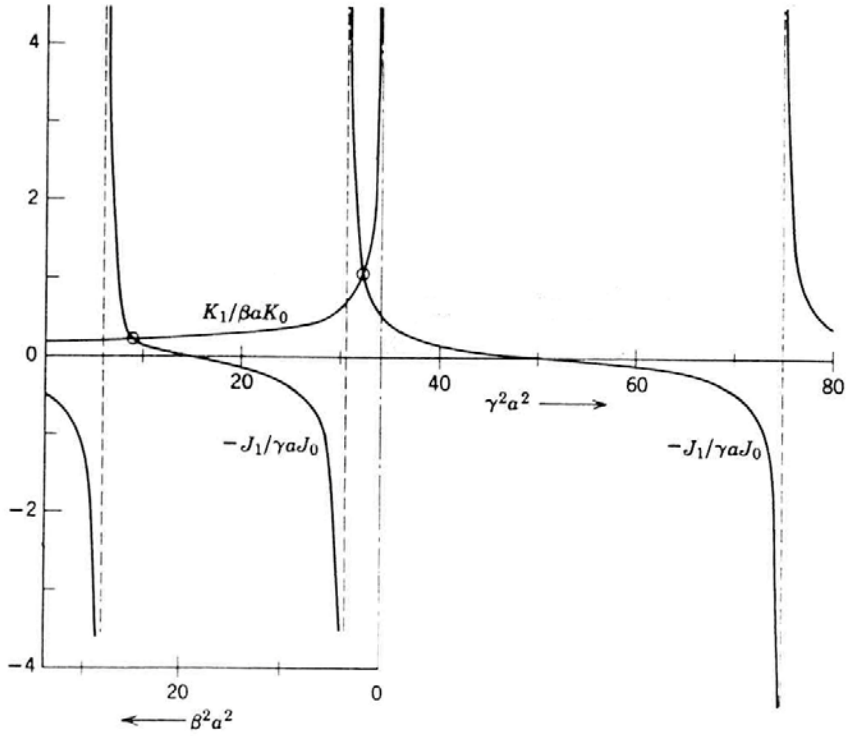
$$\gamma^2 + ß^2 = (\gamma_1\varepsilon_1 - \gamma_0\varepsilon_0)\omega^2$$

(29) With (28.2) we can finally calculate k:

$$k = \sqrt{ß^2 + \gamma_0\varepsilon_0\omega^2} = \sqrt{\gamma_1\varepsilon_1\omega^2 - \gamma^2}$$

<u>Interpretation</u>

For the visual comprehension in (Fig. 5) both terms from (28.1) are depicted as functions from $\gamma^2 a^2 \,/\, \beta^2 a^2$, bearing in mind equation (28.2) under the premise that the frequency is high enough for two crossings [3]. The vertical asymptotes are defined by the roots from $J_0(x) = 0$.



Pic. 5 [3] – Graphical interpretation of the equations (28)

If the maximal value from γa is smaller then the first root (which is approximately 2.405), both functions don't cross for real β and therefore the wave cannot exist. So the border frequency for TE-Waves equals to

(29)

$$f = \frac{2.405}{2\pi a\sqrt{\varepsilon_1\mu_1 - \varepsilon_0\mu_0}}$$

This is an important frequency. Below that frequency the dielectric rod doesn't work as a rod guide any more, but as an antenna.

For TM-waves we get on the same way the equation

$$\frac{J_1(a\gamma)}{\gamma J_0(a\gamma)} + \frac{\gamma_0\varepsilon_0}{\gamma_1\varepsilon_1}\frac{K_1(a\beta)}{\beta K_0(a\beta)} = 0$$

The value of the border frequency is the same as for TE-waves.

TEM waves doesn't exist in the dielectric rod guide.

Dielectric rod guides are used for transmission of electromagnetic waves. For transmission microwaves the dielectric rod guide isn't the best choice because of the high microwave-losses of the dielectric material.
If the height of the dielectrically cylinder has the same value like the whole multiple of the half of the border frequency the cylinder has the characteristics of a dielectric resonator. [2] [3]

Bibliography

[1] Bronstein-Semendjajew – Taschenbuch der Mathematik – BG Teubner Verlag Leipzig - 1969

[2] Yoshihiro Konishi – Microwave electronic circuit technology - Marcel Dekker, Inc. – 1998

[3] John David Jackson – Klassische Elektrodynamik – Walter de Gruyter – 1993

[4] Hans–Jochen Bartsch – Taschenbuch mathematischer Formeln – Fachbuchverlag Leipzig – 1999

[5] Notes of the course optical and microwave propagation on the University of Technology Chemnitz, summer 2001

Used Pictures

Picture 1 – the dielectric rod guide - from Yoshihiro Konishi – Microwave electronic circuit technology - Marcel Dekker, Inc. – 1998 - page 175

Figure 2 – the dielectric wave guide – from John David Jackson – Klassische Elektrodynamik – Walter de Gruyter – 1993 – page 430

Picture 3 - Graph of the Bessel function (1. sort) and of the Weber function – from Bronstein-Semendjajew – Taschenbuch der Mathematik – BG Teubner Verlag Leipzig – 1969 – page 399

Picture 4 - Graph of the imaginary Bessel Function and of the Macdonald function – from Bronstein-Semendjajew – Taschenbuch der Mathematik – BG Teubner Verlag Leipzig – 1969 – page 400

Picture 5 - Graphical interpretation of the equations – from John David Jackson – Klassische Elektrodynamik – Walter de Gruyter – 1993 – page 433